Sequential Stochastic Dominance and the Robustness of Poverty Orderings*

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Abstract

When comparing poverty across distributions, an analyst must select a poverty line to identify the poor, an equivalence scale to compare individuals from households of different compositions and sizes, and a poverty index to aggregate individual deprivation into an index of total poverty. A different choice of poverty line, poverty index or equivalence scale can of course reverse an initial poverty ordering. This paper develops sequential stochastic dominance conditions that throw light on the robustness of poverty comparisons to these important measurement issues. These general conditions extend well-known results to any order of dominance, to the choice of individual versus family based aggregation, and to the estimation of "critical sets" of measurement assumptions. Our theoretical results are briefly illustrated using data for four countries drawn from the Luxembourg Income Study data bases.

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1 Introduction

The last decades have seen considerable developments of the methods that can be used to make comparisons of welfare distributions more robust to the choice of ethical indices. Earlier work focussed on inequality measurement (Kolm (1969), Atkinson (1970), Dasgupta et al. (1973)) and social welfare (e.g., Shorrocks (1983), but the more recent literature has also pointed out that similar robustness is desirable for poverty measurement (Atkinson (1987), Foster and Shorrocks (1988a,b)), especially in the light of the uncertainty and the frequent lack of agreement regarding the setting of poverty lines.

Less attention has been granted to robustness to the choice of equivalence scales to compare the resources of households of different compositions and sizes, although considerable uncertainty and debate also surround this choice. Two assumptions are often made on the sets of admissible equivalence scales. The first assumption says that the needs of a household increase with household size. The second assumes the existence of returns to scale to household size and of public goods within the household, and thus says that household needs do not necessarily increase as fast as household size. Even if these two assumptions were not to be disputed, a multitude of equivalence scales would still respect them. Subsequent poverty comparisons would then be determined by the scale used, and two alternative choices of equivalence scales could lead to clashing poverty orderings. This sensitivity to the choice of equivalence scales is particularly important when poverty comparisons are intended to support recommendations for poverty-alleviating economic policies. The targetting of households on a contingency other than income (such as household type) is also often convenient for purposes of poverty alleviation, and this also requires confidence in the comparisons of the standards of living of individuals belonging to different household types.

This paper provides methods that enable robust poverty orderings of distributions for large classes of equivalence scales, poverty lines and poverty indices. This will help ascertain the robustness of poverty comparisons to these important measurement tools. The methods rely on criteria of multi-dimensional stochastic dominance of censored income distributions. These criteria are easy to check and have a nice graphical interpretation. The methods developed here extend the well-known Atkinson (1992) stochastic dominance criteria to any arbitrary order of dominance[1]. In cases in which first-order dominance is not achieved, this extension effectively allows an analyst to seek robust poverty orderings through focussing on smaller sets of poverty indices. We also extend the previous dominance criteria to the choice of individual versus household based aggregation (previous work has focussed on the latter type of aggregation). This is important since there are good ethical reasons for which to prefer individual as against household counts in comparing poverty across distributions. In the presence of significant demographic differences between two distributions, the choice of one or of the other type of aggregation can also impact on the robustness and on the ordering of poverty comparisons (as we find in the illustration below).

Finally, we show how sequential stochastic dominance can be used to identify "critical sets" of measurement assumptions. This latter exercise provides estimates of critical boundaries for the ranges of equivalence scales and poverty lines over which a poverty comparison may be considered robust. It also demonstrates clearly the trade-offs that arise in trying to infer robustness over sets of poverty indices, poverty lines and equivalence scales that are as large as possible. We find that the higher the order of dominance or the smaller the range of possible poverty lines for an equivalent adult, the wider the sets of equivalence scales over which robust poverty orderings can be inferred. Conversely, the smaller the sets of equivalence scales considered, the wider are the robust class of poverty indices and the robust range of poverty lines for single adults.

The rest of the paper runs as follows. Section 2 develops sequential stochastic dominance conditions in the presence of heterogeneous household composition. Section 3 develops analogous conditions for the cases in which individuals (rather than households) are the units of aggregation. When orderings are not robust to pre-specified sets of poverty indices, equivalence scales or poverty lines, methods to identify the subsets for which the orderings may be considered robust are described in Section 4. Section 5 applies our methods to data of four countries drawn from the Luxembourg Income Study. The last section concludes the paper and summarises some of our main findings.

[1]On sequential dominance, see also Atkinson and Bourguignon (1987) and Bourguignon (1989) for social welfare comparisons and Jenkins and Lambert (1993) for poverty comparisons.

2 Sequential Stochastic Dominance with Households as the Aggregating Units

By definition, a poor is someone whose income[2], y, falls under a minimum threshold, z, which we term the poverty line. To quantify the importance of individual and aggregate poverty in a distribution, one can choose from a long list of existing aggregate indices and poverty line definitions. Focussing on one specific index or on one specific poverty line can, however, lead to serious difficulties. On the one hand, it is clear that a cardinal measure of poverty depends on the particular index selected and on the chosen poverty line. On the other hand, the ordinal comparison of poverty can also be sensitive to these measurement choices.

A large part of the recent literature on poverty measurement has thus been devoted to the development of stochastic dominance conditions for robust ordinal orderings of poverty[3]. These stochastic dominance conditions are, however, usually applied to distributions of adult-equivalent incomes. This leaves open the question of how to proceed in the presence of differences in judgement regarding the choice of equivalence scales and poverty lines for households of different types. The problem is important since a number of recent studies have emphasized the sensitivity of poverty profiles to the choice of equivalence scales[4]. To deal with this issue, Atkinson (1992) and Jenkins and Lambert (1993) developed a first–order sequential stochastic dominance criterion for poverty comparisons in the presence of heterogeneity in household size. This section and the next extend this work by developing

[2]Alternatively, this could of course be utility, consumption, or other indicators of individual welfare.

[3]See for instance Atkinson (1987) and Foster and Shorrocks (1988a, b).

[4]Buhmann et al. (1987) show the important empirical impact of different equivalence scale parameters on poverty measurement. Coulter et al. (1992b) use the same type of parameterization and analyze its implication for the theoretical impact of equivalence scales on poverty measurement (see also Coulter et al. (1992a)). Duclos and Mercader-Prats (1999) generalize the analysis for a class of parameterized equivalence scales extended to take household composition into account and find that poverty profiles and orderings are sensitive to assumptions on needs. Phipps (1991) uses Canadian micro-data to show the sensitivity of poverty profiles to the choice of equivalence scales. Burkhauser et al. (1996) note that comparisons of poverty composition between Germany and the United States are strongly influenced by assumptions on household needs. Finally, De Vos and Zaidi (1997) report that comparisons of poverty between EEC countries and across different demographic subgroups are quite sensitive to the choice of equivalence scales.

sequential dominance criteria for arbitrary stochastic orders[5]. As we will see, these criteria also have the considerable advantage of providing simultaneous robustness checks with respect to the choice of poverty indices and poverty lines.

We assume in this section that it is households that are counted when it comes to computing aggregate indices of poverty (as Jenkins and Lambert (1993) and Atkinson (1992) also assume throughout[6]). The next section will show how the dominance criteria are altered when it is individuals that are counted in the aggregation of poverty. For both of these sections, we consider a class Π^s of additive poverty indices that respect some s-order dominance assumptions. Households are heterogenous in size. For expositional and analytical simplicity, however, households of the same size are assumed to be homogenous. We assume that there are n household types. We wish to determine if poverty decreases when we move from an initial income distribution A to an alternative income distribution B. Let $F_{Ak}(x)$ be the continuous distribution of the income x of the households of k individuals in distribution A. $F_{Bk}(x)$ is defined analogously. Those distribution functions are defined over $[0, a]$, where a is any income level exceeding the maximum income and the maximum conceivable poverty line for any of the household types in the two distributions. Thus, we have $F_{Ak}(a) = F_{Bk}(a) = 1$, for all k. Let θ_{Ak} represents subgroup k's population share in distribution A. Let $D^1_{Ak}(x) = F_{Ak}(x)$ and $D^s_{Ak}(x) = \int_0^x D^{(s-1)}_{Ak}(y)\,dy$, for all integer $s \geq 2$. By successive integrations, we can show that

$$D^s_{Ak}(x) = \frac{1}{(s-1)!}\int_0^x (x-y)^{s-1} dF_{Ak}(y) \tag{1}$$

which, as we will see later, has useful links with a well-known class of poverty indices. θ_{Bk} and $D^s_{Bk}(x)$ are defined analogously.

In the manner of Jenkins and Lambert (1993) and Atkinson (1992), we assume that we can agree on the following ordering of possible poverty lines across household types:

$$z_1 \leq z_2 \leq z_3 \leq \ldots \leq z_n < a, \tag{A1}$$

[5]The notation used here is similar to Jenkins and Lambert's (1993) since it facilitates the identification of the effects of differences in demographic composition.

[6]Note however that Jenkins and Lambert (1993) mention that household aggregation may not be the best aggregating procedure, and they also discuss some of the issues that could be involved in aggregating over individuals instead.

where z_k is the poverty line for households of k individuals. Also assume in the same spirit that we can agree on a set of maximum poverty lines z_k^+ for each of the different types k of households, such that

$$z_k \leq z_k^+, \forall k = 1, ..., n \tag{A2}$$

Assumptions A1 and A2 together imply that $z_1^+ \leq z_2^+ \leq z_3^+ \leq ... \leq z_n^+ < a$.

As is often done in the literature on poverty and equivalence scales, we can interpret the ratio of z_k over z_1 as the number of equivalent adults living in a household of type k. Denote by $m(k)$ the equivalence scale (relative to a household of a single adult) for such a household; we then have

$$m(k) = z_k/z_1, \text{ with } m(1) = 1. \tag{A3}$$

By assumptions A1 and A3, it follows that

$$m(k) \leq m(k+1), \text{ for } k = 1, ..., n-1. \tag{2}$$

A common and usually undisputed restriction on equivalence scales is that $m(k)$ cannot exceed k, that is, with A1 and A3, that:[7]

$$m(k) \in [1, k]. \tag{A4}$$

Assumptions A1, A2, A3 and A4 together suggest that z_k^+ may often be sensibly set as follows:

$$z_k^+ = k z_1^+ \tag{3}$$

where z_1^+ is an agreed or pre-specified maximum poverty line for households of one person.

Since the poverty indices $P \in \Pi^s$ are assumed to be additive, we can write

$$P_A = \sum_{k=1}^{n} \theta_{Ak} \int_0^a p_k(x)\, dF_{Ak}(x). \tag{A5}$$

where $p_k(x) \geq 0$ is the (finite) contribution of a household of size k and of total income x to aggregate poverty and where the focus axiom imposes

[7]The assumptions made on $m(k)$ here do not imply the assumption of concavity in k sometimes found in the literature. $m(k)$ can indeed be locally convex in k, although we may wish to assume that it cannot exceed k.

that $p_k(x) = 0 \; \forall x \geq z_k, \forall k$. P_B is defined analogously. If poverty does not increase when we move from A to B, we have that

$$\Delta P_{AB} = \sum_{k=1}^{n} \theta_{Bk} \int_0^a p_k(x)\, dF_{Bk}(x) - \sum_{k=1}^{n} \theta_{Ak} \int_0^a p_k(x)\, dF_{Ak}(x) \leq 0. \qquad (4)$$

For s-order stochastic dominance, we also assume that

$$p_k(x) \in C^s, \qquad \text{(A6)}$$

where C^s is the set of continuous functions which are s-time differentiable over $[0, a]$. Assumption A6 implies *inter alia* that an infinitesimal increase in the income x of a household does not induce a significant variation in the function $p_k(x)$. This rules out at present the popular poverty headcount, although we will return to that index later.

In order to develop sequential stochastic dominance criteria of order s, we need a final assumption:

$$(-1)^s p_n^{(s)}(x) \geq ... \geq (-1)^s p_2^{(s)}(x) \geq (-1)^s p_1^{(s)}(x) \geq 0 \qquad \text{(A7)}$$

For $s = 1$, assumption A7 implies that an increase in household income x diminishes poverty, whatever the household type to which this increased income accrues. It also says that, for a given household income x, the potential for such poverty reduction is greater for households with more members[8]. For $s = 2$, assumption A7 says that an equalising transfer of \$1 to a poor from a richer individual decreases poverty, and this effect is stronger across households of larger sizes[9]. Together, assumptions A5, A6 and A7 define the classes of poverty indices Π^s, $s = 1, 2, ...$[10]

[8]Assumption A7 only orders derivatives across household types at a given level of total household income. In particular, it does not say whether a transfer from a richer household towards a poorer household may be desirable when the rich household is composed of more individuals than the poor household. See Ebert (1997) for more discussion of this issue.

[9]The interpretation of A7 for higher s can be made using Fishburn and Willig (1984), where their general transfer principles give increasing weights to transfers occurring at the bottom of the distribution as s increases. Here, A7 makes these principles more normatively important for larger households than for smaller ones.

[10]We can show that the continuity assumption A6 together with $p_k(x) = 0 \; \forall x \geq z_k$ and assumption A7 yield $\Pi^s \subset \Pi^{s-1}$, for $s = 2, 3, ...$

At this point, it can be worth noting that the popular $FGT(\alpha)$ indices (see Foster et al. (1984))[11] defined by

$$FGT(\alpha) = \int_0^z (z-y)^\alpha \, dF(y), \tag{5}$$

satisfy assumption A7 (when A1 is also satisified) for $\alpha \geq s$. For these indices we have

$$p_k^{(s)}(x) = (-1)^s \left[\prod_{i=0}^{s-1} (\alpha - i) \right] (z_k - x)^{\alpha - s}. \tag{6}$$

We can check that $p_k^{(s)}$ is negative when s is odd and positive when s is even. Moreover, $\left| p_k^{(s)} \right|$ increases when z_k increases. Other well known additive indices also satisfy assumption A7 weakly, such as the index of Watts (1968) for which $p_k(x) = \ln(z_k) - \ln(x)$, and the index of Clark et al. (1981)[12] where $p_k(x) = \frac{1}{c}[(z_k)^c - x^c]$, $c \leq 1$. For these two indices, the derivatives of any order at a given level of income are equal for all types of households. Furthermore, the signs of the derivatives are as assumed in A7. These indices thus satisfy weakly assumptions A7.

We are now ready to state a first result (for ease of exposition, the proofs of the propositions appear in the appendix).

Proposition 1 $\Delta P_{AB} \leq 0$ *for all* P *satisfying A5, A6 and A7 and for all poverty lines satisfying assumptions A1 and A2 if*

$$\sum_{k=l}^{n} \left[\theta_{Ak} D_{Ak}^s(x) - \theta_{Bk} D_{Bk}^s(x) \right] \geq 0, \ \forall x \leq z_l^+, \ \forall l. \tag{DS}$$

When $s = 1$, condition DS is identical to the condition developed by Jenkins and Lambert (1993), and it is also similar to the one found in Atkinson (1992). When graphed against x, condition DS for a given l shows the difference between A and B in the contribution of the cumulative groups l to n to

[11] The original formulation proposed by Foster and al. (1984) involves a normalisation by z_k, which means that asumption A3 would not be satisfied for these indices. Atkinson (1992) argues, however, that such normalisation may be inappropriate in the context of heterogeneous households. Davidson and Duclos (1998) and Foster and Sen (1997) also show that a normalisation of poverty indices by different poverty lines can generate orderings that are more naturally interpretable in terms of relative inequality than in terms of poverty.

[12] Again, the original formulation proposed by Clark and al. (1981) involves a normalisation by z_k, which means that asumption A3 would not be satisfied for these indices.

total poverty, when the contribution is measured by the well-known additive decomposition of the FGT indices. Hence, to assess whether A has robustly more poverty than B, it is sufficient to determine whether the contribution of groups of larger households to overall poverty is always larger in A than in B. For a robust ordering, this condition must be satisfied whatever the groups considered ($l = 1, 2, ..., n$) and for all $z_k \leq z_k^+$.

At this point, it is worth noting that the condition to provide a robust ordering for the headcount index (a discontinuous index) in the presence of household heterogeneity can be generally more demanding than condition D1. For a headcount poverty ordering to be robust over the set of all poverty lines obeying assumptions A1 and A2, we need:

$$\sum_{k=1}^{n} \left[\theta_{Ak} D_{Ak}^{1}(u_k) - \theta_{Bk} D_{Bk}^{1}(u_k)\right] \geq 0, \tag{DHE}$$

$\forall u_k \in \left[u_{k-1}, z_k^+\right]$ for every $k = 2$ to n, and $\forall u_1 \in \left[z_1^-, z_1^+\right]$ where $z_1^- \geq 0$ is a possible lower bound for the set of poverty lines for households of one person.

If there is only one type of households, condition DHE is exactly the restricted stochastic dominance condition given in Atkinson (1987) for the headcount index:

$$D_{A1}^{1}(x) - D_{B1}^{1}(x) \geq 0 \ \forall x \in \left[z_1^-, z_1^+\right]. \tag{7}$$

When $z_1^- > 0$, condition DHE is less demanding than condition D1, since D1 is given by

$$D_{A1}^{1}(x) - D_{B1}^{1}(x) \geq 0 \ \forall x \in \left[0, z_1^+\right]. \tag{8}$$

which is the unrestricted first-order poverty dominance condition also found in Atkinson (1987).

If we add a second type of households, DHE becomes

$$\begin{gathered} \theta_{A1} D_{A1}^{1}(u_1) - \theta_{B1} D_{B1}^{1}(u_1) + \theta_{A2} D_{A2}^{1}(u_2) - \theta_{B2} D_{B2}^{1}(u_2) \geq 0, \\ \forall u_2 \in \left[u_1, z_2^+\right], \forall u_1 \in \left[z_1^-, z_1^+\right]. \end{gathered} \tag{9}$$

For every income level u_1 for a single person, we need to test the above dominance condition for every income level from u_1 to z_2^+ for households of two persons. For condition D1, however, we would only need to test the dominance condition at common levels of household income x.

If we add a third type of households, dominance must be tested for every combination of income levels $u_1 \in \left[z_1^-, z_1^+\right]$, $u_2 \in \left[u_1, z_2^+\right]$ and $u_3 \in \left[u_2, z_3^+\right]$

This requirement illustrates how DHE becomes more and more demanding relative to DS as the number of household types rises. Headcount-based poverty orderings can thus be expected to be generally less robust to the choice of equivalence scales than continuous poverty indices (thus possibly adding to the well-known shortcomings of this index).

Conditions DS and DHE also make clear that robust poverty reductions can be achieved either through a change in the distribution of income within a group, through a redistribution of income across groups, or through a change in household demographic composition (holding the groups' distributions of incomes constant). Jenkins and Lambert (1993) distinguish between the effects of income distribution changes and the effects of demographic changes by suggesting the following reformulation of condition D1:

$$\sum_{k=l}^{n} \theta_{Ak} \Delta D_k^1(x) + \sum_{k=l}^{n} \Delta\theta_k D_{Bk}^1(x) \geq 0, \ \forall x \leq z_l^+, \ \forall l, \tag{D1'}$$

where $\Delta D_k^1(x) = D_{Ak}^1(x) - D_{Bk}^1(x)$ and $\Delta\theta_k = \theta_{Ak} - \theta_{Bk}$. This equation shows how income distribution changes, $\Delta D_k^1(x)$, may be reinforced or offset by demographic changes, $\Delta\theta_k$. However, we cannot interpret equation D1' exactly as the sum of an income distribution change effect and a demographic change effect since the term $\Delta\theta_k$ is really in joint interaction with $\Delta D_k^1(x)$. Instead, we can more generally rewrite DS in the spirit of a decomposition of the FGT index proposed by Ravallion and Huppi (1991):

$$\sum_{k=l}^{n} \theta_{Ak} \Delta D_k^s(x) + \sum_{k=l}^{n} \Delta\theta_k D_{Ak}^s(x) + \sum_{k=l}^{n} \Delta\theta_k \Delta D_k^s(x) \geq 0, \ \forall x \leq z_l^+, \ \forall l, \tag{DS''}$$

where $\Delta D_k^s(x) = D_{Ak}^s(x) - D_{Bk}^s(x)$. This is more precisely interpreted as the sum of an income distribution change effect, a population change effect and the interraction between the two effects.

3 Sequential Stochastic Dominance with Individuals as the Aggregating Units

Households were assumed in the last section to be the appropriate aggregating units when it came to capture overall poverty into the aggregate poverty indices P. Variable θ_{Ak} represented the ratio of the number of households of

k individuals to the total number of households in distribution A. Ultimately, however, it is generally the welfare of individuals that matter for normative purposes, and it is then ethically preferable to count individuals rather than households in comparing poverty. (Household formation matters in so much as it influences the standards of living of the household members.) For this reason, we now derive the appropriate dominance criteria when individuals are the aggregating units[13].

As before, we suppose that the poverty indices $P \in \Pi^s$ are additive and that there are n different sizes of household. We now denote by $F_{Ak}(y)$ the distribution function of the *per capita income* of households of k individuals in distribution A. It total household income is x, per capita income is x/k. These distribution functions are defined on an the interval $[0, a]$, where a is greater than the maximum per capita income for all types of households and is also greater than the maximum conceivable poverty line (in per capita terms) for all types of individuals. Thus, we have $F_{Ak}(a) = 1$, for all k. We will denote by γ_{Ak} the population proportion of individuals who are members of households of k individuals in A . It then follows that:

$$\gamma_{Ak} = \frac{k\theta_{Ak}}{\sum_{l=1}^{n} l\theta_{Al}}. \tag{10}$$

As before, let us also define $D_{Ak}^1(y) = F_{Ak}(y)$ and $D_{Ak}^s(y) = \int_0^y D_{Ak}^{(s-1)}(u)\,du$ for all integers $s \geq 2$. $F_{Bk}(y)$, γ_{Bk} and $D_{Bk}^s(y)$ are all defined analogously.

We now assume that the individual poverty lines z_k are ranked as follows in ascending order of individual needs:

$$a > z_1 \geq z_2 \geq z_3 \geq ... \geq z_n, \tag{A8}$$

Assumptions A8 and A2 together imply that $a > z_1^+ \geq z_2^+ \geq z_3^+ \geq ... \geq z_n^+$.

An implicit equivalence scale $m(k)$ (to transform the total income of a household of k individuals into an equivalent income for a one-person house-

[13]Jenkins and Lambert (1993) argue that counting individuals for the aggregation of poverty raises the problem of how to infer hidden individual standards of living from observed total household income. The method frequently used to tackle this problem supposes that the household's equivalent income is enjoyed identically by all household members (an assumption that we also use here). Although this assumption is clearly open to criticism, it is in our view as relevant and as problematic to the household-aggregating approach (which relies on an average standard of living for the household) as for the individual-aggregating approach (which assumes equal sharing).

hold) is now given by:

$$m(k) = k \cdot z_k / z_1, \text{ with } m(1) = 1. \tag{A3I}$$

Sensible bounds for $m(k)$ are again given by assumption A4, and by assumption A3I, the maximum poverty line for individuals in household type k can be set to:

$$z_k^+ = z_1^+. \tag{11}$$

Knowing that every $P \in \Pi^s$ is additive, we can again write

$$P_A = \sum_{k=1}^{n} \gamma_{Ak} \int_0^a p_k(y)\, dF_{Ak}(y). \tag{A9}$$

where $p_k(y) \geq 0$ is the (finite) contribution to overall poverty of per capita income y for an individual who is a member of a household type k, and with $p_k(y) = 0$, $\forall y \geq z_k, \forall k$. There will be no increase in poverty when we move from distribution A to distribution B if

$$\Delta P_{AB} = \sum_{k=1}^{n} \gamma_{Bk} \int_0^a p_k(y)\, dF_{Bk}(y) - \sum_{k=1}^{n} \gamma_{Ak} \int_0^a p_k(y)\, dF_{Ak}(y) \leq 0. \tag{12}$$

We make for $p_k(y)$ the same continuity assumption as for $p_k(x)$ in the previous section:

$$p_k(y) \in C^s, \tag{A10}$$

The final assumption for individual-based poverty indices is:

$$(-1)^s p_1^{(s)}(y) \geq (-1)^s p_2^{(s)}(y) \geq ... \geq (-1)^s p_n^{(s)}(y) \geq 0. \tag{A11}$$

For $s = 1$, this again implies that an increase in household per capita income y diminishes poverty, whatever the household type to which the individual belongs. It also says that, for a given per capita income y, the potential for such poverty reduction is greater for individuals living in smaller households[14]. For $s = 2$, assumption A11 now says that an equalising transfer of \$1 of per capita income to a poor from a richer individual decreases poverty, and that this effect is stronger for individuals belonging to households of

[14]It is in smaller households that economies of scales and the opportunities for sharing public goods are presumably the smallest. It is therefore also for members of smaller households that, for a given per capita income, individual welfare is lower.

smaller sizes[15,16]. Together, assumptions A9, A10 and A11 define the classes Π^s of poverty indices for the individual aggregation approach.

We can now state the following general result.

Proposition 2 $\Delta P_{AB} \leq 0$ *for all poverty indices P satisfying A9, A10 and A11 and for all poverty lines satisfying assumptions A2 and A8 if*

$$\sum_{k=1}^{l} \left[\gamma_{Ak} D^s_{Ak}(y) - \gamma_{Bk} D^s_{Bk}(y)\right] \geq 0, \ \forall y \leq z_l^+, \ \forall l. \qquad \textit{(DIS)}$$

Although the approach using households as the aggregating unit (call it approach H) is analytically similar to that using individuals (approach I, say), they are not identical and do not necessarily generate the same poverty orderings. Three reasons account for this.

First, approach I counts individuals rather that households and thus gives a higher ethical weight to members of larger households. Second, assumptions A7 and A11 on the rankings of the successive derivatives reverse the ordering of "needs". This can have immediate effects if, for example, in approach I dominance for individuals in households of one person initially fails, or alternatively if in approach H dominance for individuals in households of n persons initially fails.

A numerical example might help to illustrate these differences. Suppose that a distribution is made up of twenty households. Ten of these households are made of couples whereas the ten others are composed of single people. Also, suppose that the maximum poverty line for a single person, z_1^+, is 3. In A, the incomes of the single people are $\{2, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the total incomes of the couples are $\{2, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In B, the incomes of the single people are given by $\{2, 2, 2, 4, 5, 6, 7, 8, 9, 10\}$ and those of the couples are $\{4, 4, 5, 5, 6, 6, 7, 8, 9, 10\}$. Comparing A and B, we find that poverty increases for single people and decreases for couples. How, however, is overall poverty affected by the movement of A to B? Consider first the H approach

[15] For a given difference in per capita income, the difference in needs-adjusted standards of living is presumably larger among individuals of smaller households (for reasons discussed in the previous footnote), and hence it is there that redistributive effects from Dalton transfers are presumably the greatest.

[16] Again, several indices used in the literature obey this condition, as can be seen from the discussion of (6) in the above section by switching the signs in assumption A7 and by reversing the orderings of the z_k.

to aggregation for first-order dominance (condition D1 with $n = 2$.).We note that since

$$\frac{1}{2}D^1_{A1}(2) - \frac{1}{2}D^1_{B1}(2) = \frac{2}{20} - \frac{3}{20} = \frac{-1}{20} \tag{13}$$

is compensated by

$$\frac{1}{2}D^1_{A2}(2) - \frac{1}{2}D^1_{B2}(2) = \frac{2}{20} - \frac{0}{20} = \frac{2}{20} \tag{14}$$

we have

$$\sum_{k=1}^{2}\left[\theta_{Ak}D^1_{Ak}(2) - \theta_{Bk}D^1_{Bk}(2)\right] = \frac{2}{20} + \frac{-1}{20} = \frac{1}{20} \geq 0.$$

Since the distributions are otherwise identical, we can infer robustness of overall poverty reduction when moving from A to B.

Consider now approach I to aggregating poverty. Condition DI1 says that it is necessary first to find dominance for single individuals. The increase in poverty for single persons (see (13)) can then not be compensated by the reduction in poverty for couples (see (14)). Within such a framework, we cannot infer robustness of overall poverty reduction when moving from A to B.

Finally, the income levels at which households and individuals are compared are not the same for approaches H and I. Approach H applies the following conditions

$$\sum_{k=l}^{n}\left[\theta_{Ak}D^s_{Ak}(x) - \theta_{Bk}D^s_{Bk}(x)\right] \geq 0, \tag{15}$$

by using total household income x whatever the type of households. Approach I applies the following conditions

$$\sum_{k=1}^{l}\left[\gamma_{Ak}D^s_{Ak}(y) - \gamma_{Bk}D^s_{Bk}(y)\right] \geq 0, \tag{16}$$

by using per capita income y $(= x/k)$ regardless of household size. A numerical example will help illustrate this difference. Suppose distributions A and B are each made up of thirty households. Ten of these households are couples with a child, ten others are couples without children, whereas the last ten are single people. Let us suppose again that the maximum poverty line for a single person, z_1^+, is 3. In A, each type of households has an identical distribution of income which is $\{2, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In B, singles and couples with a child have incomes $\{3, 3, 3, 4, 5, 6, 7, 8, 9, 10\}$ while those of the couples without children are $\{2, 2, 2, 4, 5, 6, 7, 8, 9, 10\}$. We thus note a poverty reduction for singles and couples with a child but a poverty increase for couples without children. How is overall poverty affected by the movement of A to B? On the one hand, with approach H, we note that

$$\frac{1}{2}D^1_{A1}(2) - \frac{1}{2}D^1_{B1}(2) = \frac{2}{20} - \frac{0}{20} = \frac{2}{20}$$

and

$$\frac{1}{2}D^1_{A3}(2) - \frac{1}{2}D^1_{B3}(2) = \frac{2}{20} - \frac{0}{20} = \frac{2}{20}$$

compensate for

$$\frac{1}{2}D^1_{A2}(2) - \frac{1}{2}D^1_{B2}(2) = \frac{2}{20} - \frac{3}{20} = \frac{-1}{20}.$$

Thus, we have dominance. On the other hand, with approach I, condition DI1 stipulates that the loss for individuals within couples without children must be compensated by the gain for the single people. Here, the gain of 2/20 among single people would indeed compensate for the loss of 1/20 for the couples without children if the comparisons were done at total household income (and this, even when taking into account that $\gamma_{A2} = 2 \cdot \gamma_{A1}$). With DIS, comparisons are rather made at common levels of per capita incomes, and the loss of 1/20 for couples without children is at a per capita income of \$1. Because there is no compensating gain at this per capita income for singles, we cannot conclude to a robust decrease in poverty with the I approach.

4 Assessing the Robustness of Poverty Orderings

When conditions DS or DIS are met, we may confidently assert that poverty in B is no greater than poverty in A for all of the additive poverty indices that satisfy the continuity and derivative assumptions described above, and for all of the poverty lines satisfying A1 and A2 or A2 and A8. As indicated earlier, it may then be useful to consider z_1^+ as an upper bound for the poverty line of single persons, and to interpret the ratios of z_k^+/z_1^+ or kz_k^+/z_1^+as indicative of the bounds of the ranges of equivalence scales over which the poverty robustness result holds.

Now suppose that the test of conditions DS or DIS has failed and that we therefore cannot infer a robust poverty ordering over an initially specified set of poverty indices and poverty line upper bounds z_k^+. One option would be simply to conclude that the distributions cannot be robustly ranked. Three alternative routes can however be followed to determine whether dominance can be secured over smaller subsets of poverty indices and poverty lines than initially envisaged. The first route increases the order of stochastic dominance until a poverty ordering becomes robust over all of the pre-specified ranges of poverty lines. The second route infers (for a given s) critical bounds for restricted intervals of poverty lines for single persons while maintaining the ranges of the ratios of z_k^+/z_1^+. Finally, the third route determines the critical ratios of z_k^+/z_1^+ up to which a poverty ordering is robust for a given poverty line upper bound z_1^+ for single people and for a pre-specified order of stochastic dominance s.

A possible empirical strategy then runs as follows[17]. In order to determine if poverty is unambiguously higher in A than in B, we can initially test for first-order sequential stochastic dominance using condition D1 or DI1. In that first step, we may set the z_k^+ as we wish (so long as assumptions A1 and A2 or assumptions A2 and A8 are met), but, as mentioned above, sensible bounds are arguably given by equations (3) and (11). If condition D1 or DI1 does not lead to an unambiguous ranking, we conclude that it is not possible to obtain a poverty ordering of the two distributions which is robust to all members of $\Pi^1$and over the pre-specified ranges of poverty lines determined by $z_1^+, z_2^+, z_3^+, ...$

A second step may then test successively conditions D2, D3,... or DI1, DI2,... to determine if there is a smaller class Π^s of poverty indices for which the ranking of these two distributions is robust. Alternatively (or simultaneously), we may choose not to restrict unduly the class of poverty indices for which the ranking of the two distributions will eventually be robust. Instead, we may prefer to restrict the interval of admissible poverty lines for single persons, while keeping constant the "reasonable" upper bounds for the equivalence scales (using restrictions (3) and (11)). Hence, for stochastic

[17]As should become clear, this strategy can be followed either with the H or the I aggregating approach.

dominance of order s, we can search for a critical z_1^+ such that z_1^+ is the maximum value of ξ which obeys the condition

$$\sum_{k=l}^{n}\left[\theta_{Ak}D_{Ak}^{s}\left(x\right)-\theta_{Bk}D_{Bk}^{s}\left(x\right)\right]\geq 0,\ \forall x\leq l\xi,\ \forall l, \tag{17}$$

for the H aggregating approach, and the condition

$$\sum_{k=1}^{l}\left[\gamma_{Ak}D_{Ak}^{s}\left(y\right)-\gamma_{Bk}D_{Bk}^{s}\left(y\right)\right]\geq 0,\ \forall y\leq \xi,\ \forall l, \tag{18}$$

for the I aggregating approach.

Finally, we may choose to restrict the intervals of the implicit admissible equivalence scales $m(k)$. This will limit the values of $z_2^+, z_3^+, ...$ without necessarily reducing z_1^+[18]. To follow this third route, we find estimates z_k^s, $k = 1, ..., n$, of the maximum poverty lines z_k^+ for which a stochastic dominance condition of order s, DS or DIS, allows a robust ordinal ranking of poverty. With the H approach to aggregation, we find, in a first step, the values of $\widehat{z}_l^s$, $l = 1, ...n$, such that $\widehat{z}_l^s$ is the maximum value of ξ which respects the condition

$$\sum_{k=l}^{n}\theta_{Ak}D_{Ak}^{s}\left(x\right)\geq\sum_{k=l}^{n}\theta_{Bk}D_{Bk}^{s}\left(x\right),\ \forall x\leq\xi. \tag{19}$$

[18]The method used here is analogous in spirit to that proposed by Lanjouw and Ravallion (1995). They want to determine if large households (with n individuals) experience more poverty than smaller ones (for example, single people). By using the parametric equivalence scale

$$m\left(n\right)=n^{\sigma}$$

proposed by Buhmann et al (1987), they then estimate the maximum elasticity σ for which the classification is robust when one uses either a given poverty index or some of the stochastic dominance conditions stated in Atkinson (1987). Our approach is significantly different since it makes it possible to compare two distributions for which household types can vary within each distribution. Moreover, we impose smaller restrictions on the shape of the sets of allowable equivalence scales since these are not restricted to belong to parametric classes of equivalence scales of the type proposed by Buhmann et al. (1987).

In the same spirit, Bradbury (1997) determines the intervals of equivalence scales for which comparisons of poverty are robust. Although not using a parametric form for the sets of equivalence scales, his method imposes an assumption of concavity on the function $m\left(n\right)$ that is not needed here. See also Fleurbaey et al. (1998), who propose sequential second-order dominance tests that assume lower and higher bounds for equivalence scale intervals.

This gives us a set of upper poverty line bounds $\widehat{z}_1^s, \widehat{z}_2^s, ..., \widehat{z}_n^s$ that may or may not obey the assumptions made on the rankings of the z_k^+ and on the ranges of equivalence scales. To ensure that $z_k^s \leq z_{k+1}^s$ and that $m(k) \leq m(k+1)$, we proceed by iteration, first by defining $z_n^s = \widehat{z}_n^s$, and then by setting the remaining z_k^s as follows:

$$z_k^s = \min\left(\widehat{z}_k, z_{k+1}^s\right), \text{ for } k = 1, ..., n-1 \tag{20}$$

Interpreting z_1^s as the robust upper bound for the poverty line of a single person, we may then use the estimated vector $z^s = (z_1^s, z_2^s, ..., z_n^s)$ to estimate the sets of equivalence scales for which a poverty ranking is robust at order s. This "critical" set of equivalence scales is given by $m(k) \in [1, z_k^s/z_1^s]$ with the additional conditions that $m(k) \leq m(k+1)$, for $k = 1, ..., n-1$. If we also wish (although we do not need) to ensure that $m(k) \leq k$, then we simply use instead the sets $m(k) \in [1, \min(k, z_k^s/z_1^s)]$. Finally, if a maximum bound z_1^+ for the range of poverty lines for single people were to be agreed a priori, and if it were the case that $z_1^+ < z_1^s$, the robust set of equivalence scales could be further extended, simply by using $\left[1, \min(k, z_k^s/z_1^+)\right]$ instead of $[1, \min(k, z_k^s/z_1^s)]$.

To follow instead the I approach to aggregation, we mostly proceed as above by searching for estimates of z_k^s, $k = 1, ..., n$, of the maximum poverty lines z_k^+ for which a poverty ordering is robust. We find, in a first step, the values of $\widehat{z}_l^s$, $l = 1, ...n$, such that $\widehat{z}_l^s$ are the maximum values of ξ which respect the condition

$$\sum_{k=1}^{l} \gamma_{Ak} D_{Ak}^s(y) \leq \sum_{k=1}^{l} \gamma_{Bk} D_{Bk}^s(y), \ \forall y \leq \xi. \tag{21}$$

Again, this gives us a set of upper poverty line thresholds $\widehat{z}_1^s, \widehat{z}, ..., \widehat{z}_n^s$ that may or may not obey the assumptions made on the rankings of the z_k^+ for the I approach and on the (sensible) ranges of equivalence scales. To ensure that $kz_k^s \leq (k+1)z_{k+1}^s$ and thus that $m(k) \leq m(k+1)$, we proceed by iteration first by defining $z_n^s = \widehat{z}_n^s$ and then by setting the remaining z_k^s as follows:

$$z_k^s = \min\left(\widehat{z}_k^s, \frac{(k+1)}{k} z_{k+1}^s\right) \tag{22}$$

Just as before, we may use the vector $z^s = (z_1^s, z_2^s, ..., z_n^s)$ to determine the set of equivalence scales for which the ranking of poverty between two distributions is robust at dominance order s. This set is such that $m(k) \in [1, kz_k^s/z_1^s]$ with the additional condition that $m(k) \leq m(k+1)$, for $k = 1, ..., n-1$. If we also wish to ensure that $m(k) \leq k$, then we may use instead the sets $m(k) \in [1, \min(k, kz_k^s/z_1^s)]$. Moreover, if we were to agree that the maximum poverty line for single people could not exceed z_1^+, with $z_1^+ < z_1^s$, then we could do as above and extend the robust set of equivalence scales by using $\left[1, \min(k, kz_k^s/z_1^+)\right]$ instead of $[1, \min(k, kz_k^s/z_1^s)]$.

For each of the two H and I approaches, if we feel the resulting set of robust equivalence scales is too limited at order s, we can proceed to higher order $s+1$. If there were sequential dominance for some limited ranges of z_k (for all k) at an order s, then for a given z_1^+, the sets of robust equivalence scales will necessarily become larger and larger as s increases (this follows from Lemma 1 of Davidson and Duclos (1998)).

5 Illustration

We now illustrate the previous methodological results using data on Canada, Finland, Italy and USA drawn from the Luxembourg Income Study (LIS) data sets. These countries were selected partly because 1991 data were available for them. We take household income to be disposable income (i.e., post-tax and transfer income) and we apply purchasing power parities drawn from the Penn World Tables[19] to convert national currencies into 1991 US dollars. We subdivide each population into six different types of households according to the number of people composing the households. We consider households of six or more individuals as part of the same category. All household observations are weighted by the LIS sample weights "hweight". Finally, negative incomes are set to 0.

[19]See Summers and Heston (1991) for the methodology underlying the computation of these parities, and http://www.nber.org/pwt56.html for access to the 1991 figures.

	Canada	United States	Finland	Italy
1 person	14806	15406	11158	11387
2 persons	28023	29691	22620	17822
3 persons	33659	33993	28673	24021
4 persons	37388	36346	32626	25481
5 persons	40020	36385	33675	26406
6 persons or more	41033	37042	34958	26023

Table 1: Average household incomes by household type in each country (1991 $US)

Table 1 shows average household incomes by household type for each country. American households of one, two and three people have the highest average incomes while Canada has the highest average income for the larger households. Single people have the lowest average income in Finland. For the other types of households, it is in Italy where average income is lowest.

	Canada	USA	Finland	Italy
1 person	30.1%	29.1%	36.5%	16.1%
2 persons	27.9%	29.6%	30.4%	22.9%
3 persons	16.0%	16.7%	14.1%	24.3%
4 persons	16.3%	15.0%	13.0%	25.2%
5 persons	6.9%	6.1%	4.6%	8.0%
6 or more persons	2.8%	3.5%	1.4%	3.5%

Table 2: Proportion of each household type in each country.

Table 2 shows the proportion of each household type in each country. The highest proportion of single people and households of two people are found in Finland . Italy, however, has a higher proportion of larger households.

	Canada	Finland	Italy
$\widehat{z}_1^1$	42674	7242	9496
$\widehat{z}_2^1$	$-$	20878	X
$\widehat{z}_3^1$	55156	29739	X
$\widehat{z}_4^1$	50523	33830	X
$\widehat{z}_5^1$	84428	$-$	X
$\widehat{z}_6^1$	X	$-$	5935

Table 3: Thresholds $\widehat{z}_k^1$ under which each country dominates (i.e., has less poverty than) the USA when aggregating households (1991 $US).

In a first step, poverty in each one of these countries is compared with poverty in the USA. Table 3 gives the first-order dominance thresholds $\widehat{z}_k^1$ for which each country dominates (has less poverty than) the USA when poverty aggregation uses households. A horizontal bar $(-)$ indicates that $\widehat{z}_k^1$ tends towards infinity and a X indicates that the country is initially dominated by (i.e., has more poverty than) the USA when this household type is introduced. Table 3 shows that, although Canada clearly dominates the USA for households of five people or more, the USA initially dominate Canada for households of six people or more. Thus, we cannot use sequential stochastic dominance (of any order[20]) to rank poverty robustly between these two countries. Italy cannot be ranked with the USA for the same reason. Finally, we can affirm that poverty is lower in Finland than in the USA for every poverty line lower than $7242 for households of one individual, for all equivalence scales such that $m(k) \in [1, k]$ and $m(k) \leq m(k+1)$, for $k = 1, ..., n-1$, and for every of the poverty indices belonging to class Π^1.

We now compare poverty in Finland with poverty in Canada to illustrate our methodology in greater details. For low z_k^+, the data first indicate that Finland (country B) dominates Canada (country A) for all relevant cumulative household types. Figures 1 to 3 illustrate this through curves $SlA = \sum_{k=l}^{n} \theta_{Ak} D_k^1(x)$ and $SlB = \sum_{k=l}^{n} \theta_{Bk} D_k^1(x)$ for $l = 1$ to 3. As noted above, these curves show the contribution of cumulative groups to total poverty (as measured here by the headcount index). The difference between

[20]Lemma 1 of Davidson and Duclos (1998) shows that when a distribution dominates another distribution over a range of incomes $[0, d]$ for an order of dominance s, it will eventually dominate the other distribution for arbitrarily large d as s increases to infinity. Hence, as s is increased, Canada will keep dominate the USA for households of five people or more, but the USA will keep dominate Canada for households of six people or more.

$S1A$ (Figure 1) and $S2A$ (Figure 2) then gives the contribution of households of 1 person to total poverty in A. Condition D1 says that as long as curve SlB is located under curve SlA, Finland (B) dominates Canada (A). The threshold values for the poverty lines $\widehat{z}_l^1$ are determined by the intersection of the two curves SlA and SlB.

	$s=1$	$s=2$	$s=3$	$s=4$
$\widehat{z}_1^s$	5337	7058	9043	11198
$\widehat{z}_2^s$	12841	17244	22309	27442
$\widehat{z}_3^s$	22829	30509	39739	50501
$\widehat{z}_4^s$	26145	34493	48311	–
$\widehat{z}_5^s$	8920	–	–	–
$\widehat{z}_6^s$	–	–	–	–

Table 4: Thresholds $\widehat{z}_k^s$ under which Finland dominates Canada when aggregating households (1991 $US)

Table 4 shows estimated thresholds $\widehat{z}_k^s$ for sequential stochastic dominance tests of order 1, 2, 3 and 4. To be able to say that poverty in Finland is lower than in Canada for any of the poverty indices belonging to class Π^1 and for all equivalence scales defined by assumptions A3, A4 and (2), we should fix $z_1^+ = \widehat{z}_5^1/5 = \$1784$, which is probably too low a threshold to have confidence in the ordering. For classes Π^2, Π^3 and Π^4, we may fix $z_1^+ = \widehat{z}_1^2 = \$7058$, $z_1^+ = \widehat{z}_1^3 = \$9043$ and $z_1^+ = \widehat{z}_1^4 = \$11198$ respectively, which are clearly more robust upper bounds.

	$z_1^+ = 3000$	$z_1^+ = 4000$	$z_1^+ = 5000$
$m(2)$	$[1, 2.97]$	$[1, 2.23]$	$[1, 1.78]$
$m(3)$	$[1, 2.97]$	$[1, 2.23]$	$[1, 1.78]$
$m(4)$	$[1, 2.97]$	$[1, 2.23]$	$[1, 1.78]$
$m(5)$	$[1, 2.97]$	$[1, 2.23]$	$[1, 1.78]$
$m(6)$	$[1, \infty]$	$[1, \infty]$	$[1, \infty]$

Table 5: Ranges of equivalence scales $\left[1, z_k^s/z_1^+)\right]$ for which Finland dominates Canada when aggregation is over households.

To increase the robust upper poverty line bound for single persons derived from Table 4, we can also choose to restrict the class of equivalence scales over which the Canada-Finland poverty ordering is robust for Π^1 indices. For classes Π^2, Π^3 and Π^4, restricting the intervals of equivalence scales yields no gain in robustness since for orders of dominance 2, 3 and 4, $\widehat{z}_k^s \geq k\widehat{z}_1^s$ for all household types. Table 5 shows the intervals of equivalence scales $\left[1, z_k^s/z_1^+)\right]$ for which poverty comparisons are robust over class Π^1 and up to maximum poverty lines for a single person of \$3000, \$4000 and \$5000. Figure 4 illustrates the relationship between the upper bounds z_k^s/z_1^+ of the intervals to which the equivalence scales $m(2)$, $m(3)$, and $m(4)$ must belong and the maximum poverty line for households of one individual when we restrict ourselves to indices in class Π^2. Table 5 and Figure 4 both show that the intervals of acceptable equivalence scales are increasingly restricted as the poverty line z_1^+ for singles increases. Figure 5 also illustrates how the upper limit of the interval for $m\,(2)$ changes when the maximum poverty line for singles and the order of dominance vary. Clearly, the higher the order of dominance or the lower the upper bound z_1^+, the larger the robust sets of equivalence scales.

	$s=1$	$s=2$	$s=3$	$s=4$
$\widehat{z}_1^s$	5780	7968	10508	13340
$\widehat{z}_2^s$	13027	17871	23101	28303
$\widehat{z}_3^s$	20878	26438	32591	39188
$\widehat{z}_4^s$	22153	27657	33525	39933
$\widehat{z}_5^s$	8110	28151	33391	38951
$\widehat{z}_6^s$	6535	30085	36181	42618

Table 6: Thresholds $\widehat{z}_k^s$ under which Finland dominates Canada when demographic differences are eliminated (1991 \$US)

It is interesting to break up the sequential dominance condition using equation DS" in order to determine what is due to differences in the income distribution within household types and what is due to differences in the demographic composition of the two countries. For this, we eliminate the differences in demographic composition by fixing the proportion of each household type in Finland to be equal to the Canadian proportion (giving the first term in DS"). Thereafter, a new $\widehat{z}_l^s$ is calculated; $\widehat{z}_l^s$ is the maximum value of ξ which obeys the condition

$$\sum_{k=l}^{n} \theta_{Ak} \Delta D_k^s\,(x) \geq 0,\ \ \forall x \leq \xi. \tag{23}$$

Table 6 shows those thresholds. The strict dominance of Finland on Canada observed in Table 4 for households of six or more individuals is explained in part by the difference in the demographic structure of the two countries. In Finland, the proportion of households of six individuals or more is 1.4% while in Canada, it represents 2.8%. Removing that difference makes poverty in Canada greater than in Finland only when the poverty line for households of six persons of more is no greater than \$6535.

	$s=1$	$s=2$	$s=3$	$s=4$
$\hat{z}_1^s$	4092	5891	7422	9030
$\hat{z}_2^s$	5197	6389	7885	9488
$\hat{z}_3^s$	5558	7033	8702	10482
$\hat{z}_4^s$	5548	7117	8820	10662
$\hat{z}_5^s$	5481	7065	8819	10710
$\hat{z}_6^s$	5513	7190	9060	11088

Table 7: Thresholds $\hat{z}_k^s$ under which Finland dominates Canada when aggregation is over individuals (1991 \$US).

Let us now consider the aggregation of poverty over individuals. Table 7 shows estimates of the thresholds $\hat{z}_k^s$ for dominance tests of order 1, 2, 3 and 4. To be able to say that poverty in Finland is lower than in Canada for any of the poverty indices belonging to class Π^1 and for any of the equivalence scales satisfying assumptions A3I, A4 and (2), we should fix $z_1^+ = \hat{z}_1^1 = \$4092$. For classes Π^2, Π^3 and Π^4, it is necessary to fix $z_1^+ = \hat{z}_1^2 = \$5891$, $z_1^+ = \hat{z}_1^3 = \$7422$ and $z_1^+ = \hat{z}_1^4 = \$9030$ respectively. Since the poverty line upper bounds for households of one person are always lower than the poverty line upper bounds for the other types of households, restricting the intervals of equivalence scales is not useful for increasing the maximum acceptable poverty line for single persons.

Comparing Tables 4 and 7, note also that the critical upper bound for the poverty lines of one-person households appears lower (whatever the order of dominance) when we use individuals rather than households as the aggregating units. We recall, however, that in Table 4 poverty lines are set for total household income, whereas they are set in terms of per capita income in Table 7. Moreover, we noted in the discussion of Table 4 above that for robustness over the class of equivalence scales defined by assumptions A1, A3 and (2), we needed to set $z_1^+ = \$1784$. For the I approach to aggregation, z_1^+ can be set as high as \$4092 for the same degree of equivalence scale robustness. Hence, it would appear that poverty is more robustly higher in Canada than in Finland when it is individuals rather than households that are the aggregating units. This also conforms with the results of Table 5 which indicated that it is in part the higher Canadian proportion of larger households which makes poverty higher than in Finland. Besides being arguably ethically preferable, weighting households by their size in the aggregation exercise then reinforces the effect of this demographic difference.

6 Conclusion

This paper develops methods for testing whether ordinal poverty orderings are robust over large sets of equivalence scales, poverty lines and poverty indices. The methods rely on well-known criteria of first-order multidimensional stochastic dominance and extend them to any arbitrary order of dominance, to the alternative choice of individual or household based aggregation, and to the estimation of "critical sets" of measurement assumptions. The latter exercise provides estimates of critical bounds for the sets of equivalence scales and poverty lines over which poverty comparisons may be considered robust at a given order of dominance. These estimates are useful to show the trade-offs involved in delimiting the critical sets of poverty indices, poverty lines and equivalence scales over which robustness may be inferred. Generally speaking, the higher the order of dominance or the lower the upper bound for the poverty lines of single individuals, the larger the robust set of equivalence scales over which poverty orderings may be considered robust.

The theoretical results are illustrated using data for four countries drawn from the Luxembourg Income Study data bases. The sequential dominance conditions fail to order Canada and the USA, or Italy and the USA, and this, whatever the selected order of dominance. We can confidently infer, however, that poverty is lower in Finland than in the USA for a wide range of poverty indices, poverty lines and equivalence scales. The comparison of Finland and Canada also serves to illustrate how the size of the robust intervals of equivalence scales is affected by the size of the intervals of admissible poverty lines for single persons and by the size of the class of admissible poverty indices. Poverty is also found to be more robustly higher in Canada than in Finland when it is individuals rather than households that are the aggregating units, highlighting the effect on poverty comparisons of demographic differences in household composition and of alternative approaches to assessing differential household needs.

Appendix

Proof of proposition 1. Integrating by parts the integral for subgroup k in equation assumption A5, we have

$$\int_0^a p_k(x)\, dF_{Ak}(x) = p_k(x)\, F_{Ak}(x)\big|_0^a - \int_0^a p'_k(x)\, F_{Ak}(x)\, dx. \tag{24}$$

We know that $F_{Ak}(0) = 0$ and that $p_k(0)$ is finite. Also, $F_{Ak}(a) = 1$ and, from assumption A5, we know that $p_k(a) = 0$. The first term on the right-hand side of (24) is then equal to 0. We can then rewrite equation (24) as

$$\int_0^a p_k(x)\, dF_{Ak}(x) = -\int_0^a p'_k(x)\, D^1_{Ak}(x)\, dx. \tag{25}$$

Now, let us assume for the moment that

$$\int_0^a p_k(x)\, dF_{Ak}(x) = (-1)^{(s-1)} \int_0^a p_k^{(s-1)}(x)\, D^{(s-1)}_{Ak}(x)\, dx. \tag{26}$$

Integrating by part equation (26), we get

$$\int_0^a p_k(x)\, dF_{Ak}(x) = (-1)^{(s-1)} \left\{ p_k^{(s-1)}(x)\, D^s_{Ak}(x)\Big|_0^a - \int_0^a p_k^{(s)}(x)\, D^s_{Ak}(x)\, dx \right\}. \tag{27}$$

$p_k^{(s-1)}(0)$ is finite and $D^s_{Ak}(0) = 0$. We have $p_k(x) = 0\ \forall x \geq z_k$ and we know, from assumption A6, that $p_k(x) \in C^s$. This means that $p^{(s-1)}(a) = 0$. Finally, $D^s_{Ak}(a)$ is finite. We can rewrite this equation as

$$\int_0^a p_k(x)\, dF_{Ak}(x) = (-1)^s \int_0^a p_k^{(s)}(x)\, D^s_{Ak}(x)\, dx. \tag{28}$$

When $s = 1$, equation (26) is simply equation (25) and we have shown that if (26) is true then, (28) is also true. This implies that equation (28) is true for all integer $s \geq 1$. From equation (28) and equation (4), we get

$$\Delta P_{AB} = (-1)^s \sum_{k=1}^n \left[\theta_{Bk} \int_0^a p_k^{(s)}(x)\, D^s_{Bk}(x)\, dx - \theta_{Ak} \int_0^a p_k^{(s)}(x)\, D^s_{Ak}(x)\, dx \right], \tag{29}$$

$$\Delta P_{AB} = (-1)^s \int_0^a \sum_{k=1}^n p_k^{(s)}(x) \left[\theta_{Bk} D^s_{Bk}(x) - \theta_{Ak} D^s_{Ak}(x) \right] dx. \tag{30}$$

Using assumptions A1, A2 and A7 and Abel's Lemma[21], it is sufficient for $\Delta P_{AB} \leq 0$ under the conditions of the proposition that $\sum_{k=l}^{n} \left[\theta_{Ak} D_{Ak}^{s}(x) - \theta_{Bk} D_{Bk}^{s}(x)\right] \geq 0 \ \forall x \leq z_l^{+}, \forall l.$ ■

Proof of proposition 2. Integrating by parts equation (12), we obtain

$$\int_0^a p_k(y)\, dF_{Ak}(y) = p_k(y)\, F_{Ak}(y)\big|_0^a - \int_0^a p_k'(y)\, F_{Ak}(y)\, dy. \tag{31}$$

We know that $F_{Ak}(0) = 0$ and $p_k(0) > 0$ if $z_k > 0$. $F_{Ak}(a) = 1$ and, from assumption A6, we know that $p_k(a) = 0$. The first term on the right side of the equation is then equal to 0. We can rewrite this equation as

$$\int_0^a p_k(y)\, dF_{Ak}(y) = -\int_0^a p_k'(y)\, D_{Ak}^{1}(y)\, dy. \tag{32}$$

Let now assume that

$$\int_0^a p_k(y)\, dF_{Ak}(y) = (-1)^{(s-1)} \int_0^a p_k^{(s-1)}(y)\, D_{Ak}^{(s-1)}(y)\, dy. \tag{33}$$

Integrating by part equation (33), we obtain

$$\int_0^a p_k(y)\, dF_{Ak}(y) = (-1)^{(s-1)} \left\{ p_k^{(s-1)}(y)\, D_{Ak}^{s}(y)\Big|_0^a - \int_0^a p_k^{(s)}(y)\, D_{Ak}^{s}(y)\, dy \right\}. \tag{34}$$

$p_k^{(s-1)}(0)$ is finite and $D_{Ak}^{s}(0) = 0$. We have $p_k(y) = 0 \ \forall y \geq z_k$ and we know, from assumption A10, that $p_k(y) \in C^s$. This means that, for $a > z_k$, $p_k^{(s-1)}(a) = 0$. Finally, $D_{Ak}^{s}(a)$ is finite. We can rewrite this equation as

$$\int_0^a p_k(y)\, dF_{Ak}(y) = (-1)^{s} \int_0^a p_k^{(s)}(y)\, D_{Ak}^{s}(y)\, dy. \tag{35}$$

Equation (32) respects the relation depicted by equation (33). We have shown that if (33) is true then (35) is also true. This imply that equation (35) is true for all integer $s \geq 1$. From equation (35) and equation (12), we obtain

$$\Delta P_{AB} = (-1)^{s} \sum_{k=1}^{n} \left[\gamma_{Bk} \int_0^a p_k^{(s)}(y)\, D_{Bk}^{s}(y)\, dy - \gamma_{Ak} \int_0^a p_k^{(s)}(y)\, D_{Ak}^{s}(y)\, dy\right], \tag{36}$$

[21] Abel's Lemma is proved in Jenkins and Lambert (1993):

Abel's Lemma: *If $x_n \geq x_{n-1} \geq \ldots \geq x_2 \geq x_1 \geq 0$, a sufficient condition for $\sum_{i=1}^{n} x_i y_i \geq 0$ is $\sum_{i=j}^{n} y_i \geq 0$ for each j. If $x_n \leq x_{n-1} \leq \ldots \leq x_2 \leq x_1 \leq 0$, the same condition is sufficient for $\sum_{i=1}^{n} x_i y_i \leq 0$.*

$$\Delta P_{AB} = (-1)^s \int_0^a \sum_{k=1}^{n} p_k^{(s)}(y) \left[\gamma_{Bk} D_{Bk}^s(y) - \gamma_{Ak} D_{Ak}^s(y)\right] dy. \tag{37}$$

Using asumption A2, A8 and A11 and Abel's Lemma, it is sufficient for $\Delta P_{AB} \leq 0$ that $\sum_{k=1}^{l} \left[\gamma_{Ak} D_{Ak}^s(y) - \gamma_{Bk} D_{Bk}^s(y)\right] \geq 0 \ \forall y \leq z_l^+, \ \forall l.$ ■

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Figure 1
Headcounts in Finland and Canada

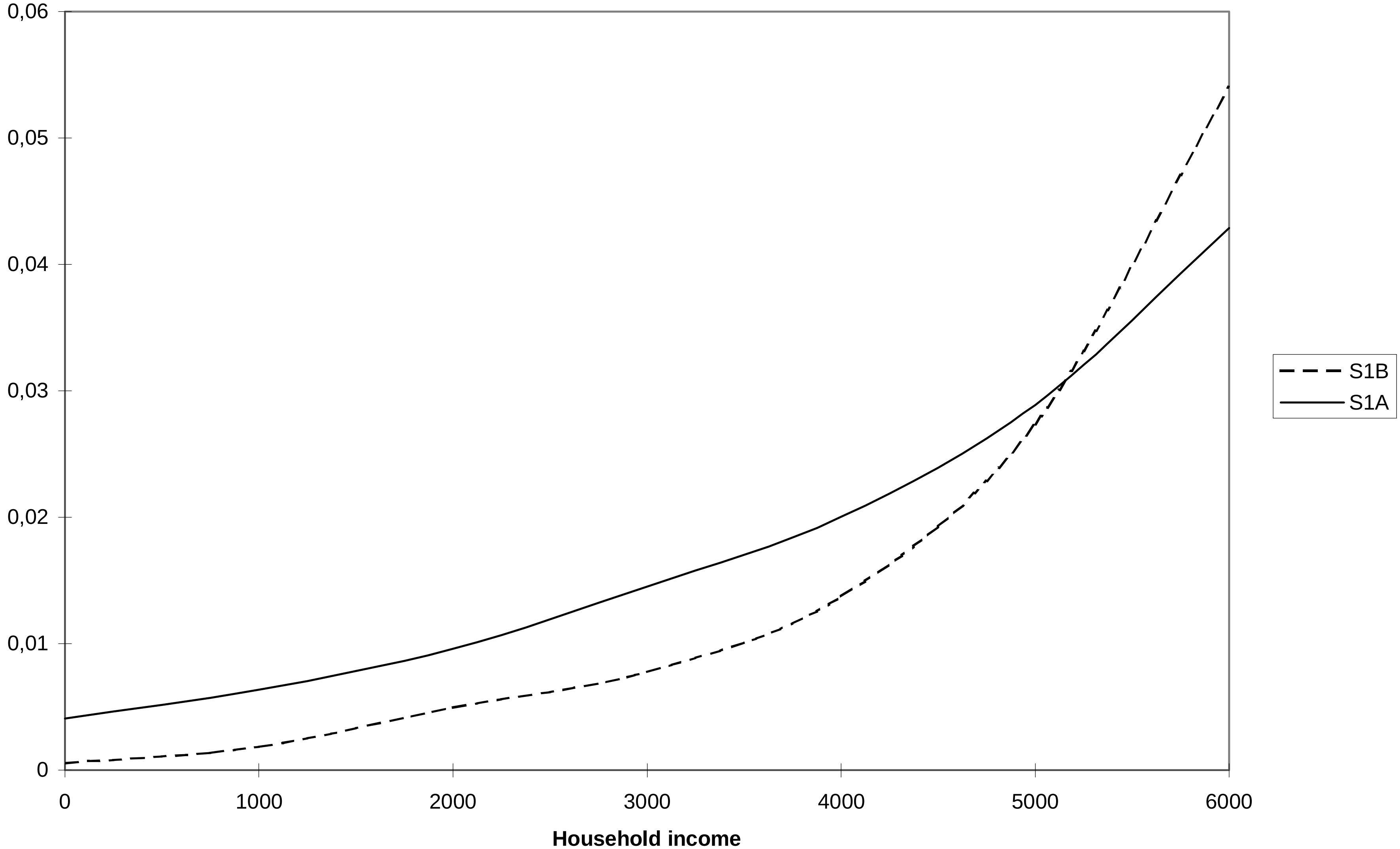

Figure 2
The contribution of households of 2 persons and more to the poverty headcount

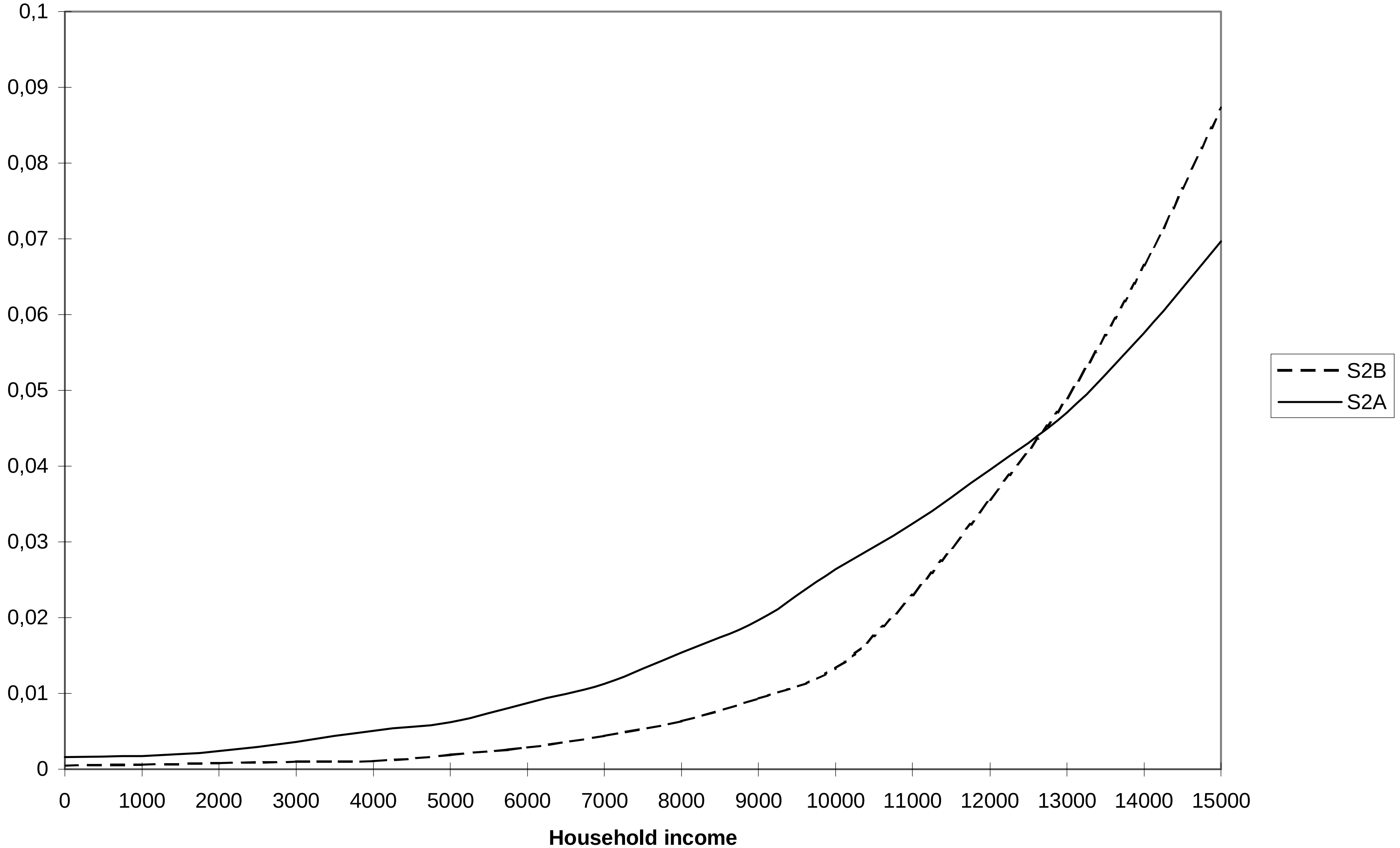

Figure 3
The contribution of households of 3 persons and more to the poverty headcount

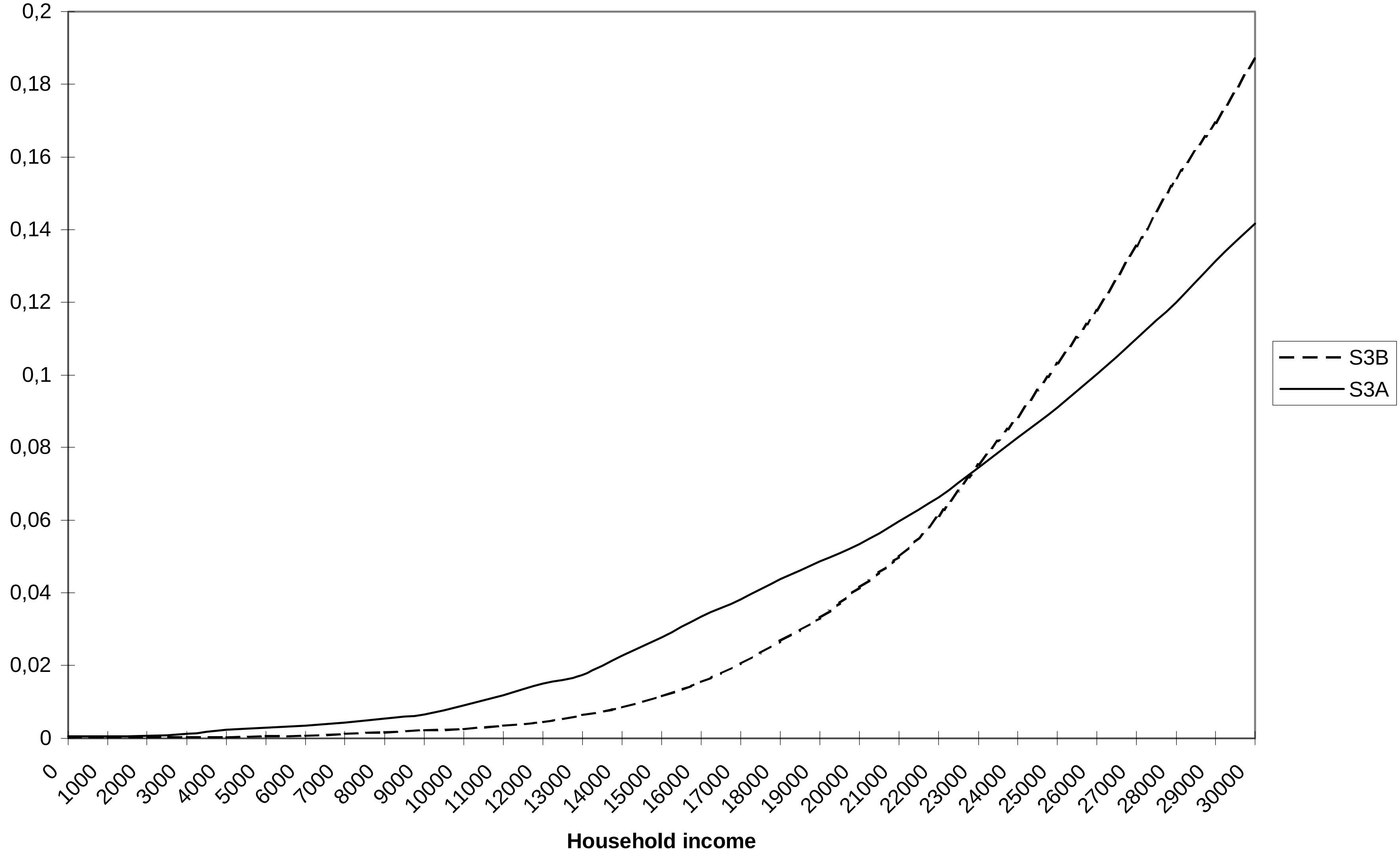

Figure 4
Equivalence scales and maximum poverty line for singles (second-order dominance)

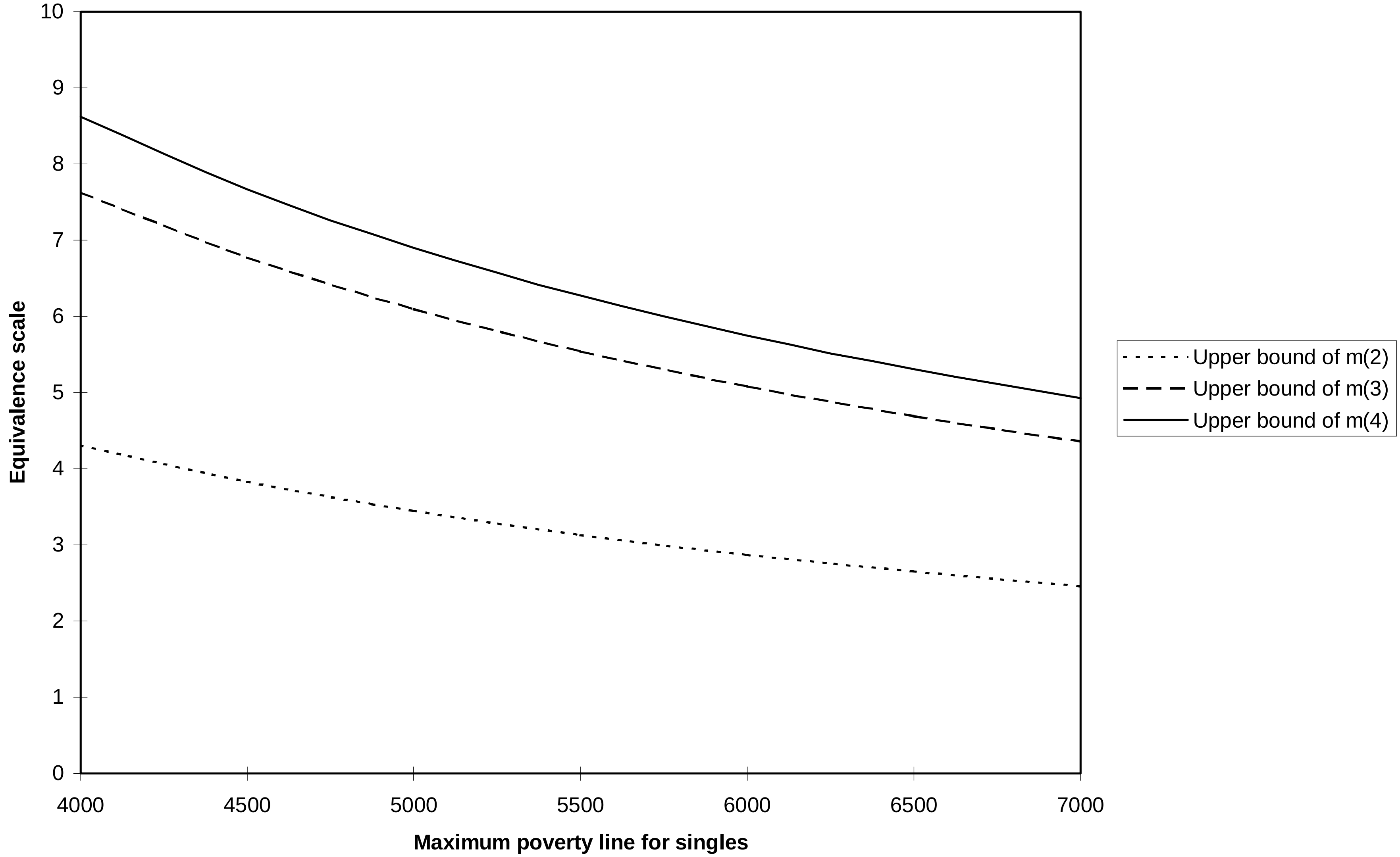

Figure 5
Equivalence scales, order of dominance and maximum poverty lines for singles